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# FUN SHARPENS THE MIND

## How simple problems bring us closer to complex mathematical theories

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This article presents a historical approach to recreational mathematics and the kinds of questions it deals with. Here, I present some relevant authors, their work, the problems they contributed to, and new fields of knowledge to which they gave rise. Starting from Egyptian mathematics, through to the Renaissance and Scientific Revolution of the 19th century, I have also devoted a special section to the work of Leonhard Euler. I have presented several problems but not the solutions, so that the readers can actively participate by solving them or, if they wish, can check the solutions in the references provided.

Keywords: **mathematics**, **recreational science**, **puzzles**, **problems**, **ingenuity**.

### INTRODUCTION: WHAT IS RECREATIONAL MATHEMATICS?

> Recreational mathematics is not easily defined because it is more than mathematics done as a diversion or playing games that involve mathematics. Recreational mathematics is inspired by deep ideas that are hidden in puzzles, games, and other forms of play. (Mathematical Association of America, 2023)

This is a possible definition of recreational mathematics published by SIGMA-Rec, the Mathematical Association of America Special Interest Group in Recreational Mathematics, which seems to be quite accurate. While we can all think of things related to recreational mathematics, a proper definition of the subject is very difficult. In this hodgepodge of elements or ideas associated with recreational mathematics, many objects have become classics, such as the Towers of Hanoi, Rubik's Cube, mathematical card games, or tessellation problems. Indeed, just as this article was about to be submitted to the editor, two discoveries were made that could be considered part of this limitless field of recreational mathematics. The first was the design of a single tile that can cover a plane to form a non-periodic mosaic. The second was the discovery of the 9th Dedekind number, a problem related to colouring the vertices of a cube analogue in different dimensions.

**«It is impossible to cover all the topics comprising the study of recreational mathematics because the earliest traces go back thousands of years»**

In this introductory article to the discipline, it is impossible to cover all the topics comprising the study of recreational mathematics because the earliest traces go back thousands of years. In fact, carved stones from 2700 BCE have been found representing the five regular polyhedra; Sumerian tablets with geometric progressions from 2400 BCE have been recorded; and the Rhind Papyrus, dated around 1650 BCE, presents a problem that is still very familiar today: a village had seven houses, each house had seven cats, each cat ate seven mice, each mouse ate seven ears of spelt, and each ear of spelt could yield seven *hekats* of grain. Houses, cats,

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mice, ears of spelt, and measures of grain... How many of these were there in the village altogether?

The Rhind Papyrus is the fundamental piece of Egyptian mathematics that has survived to the present day and contains a compendium of arithmetic and geometry, as well as a series of mathematical problems. Among these, problem 97, the one we have just mentioned, was different: for no apparent reason because it was worded in a way now typical of recreational mathematics, one that motivates the reader to think about it.

**«Many objects associated with recreational mathematics have become classics, such as the Towers of Hanoi, Rubik's Cube»**

## *DE VIRIBUS QUANTITATIS*, BY LUCA PACIOLI

Luca Pacioli (1445–1517) did a great job of collecting, systematising, and disseminating the mathematics known at the beginning of the Renaissance. Indeed, his *Summa de arithmetica, geometria, proportioni et proportionalita* ('Summary of arithmetic, geometry, proportions, and proportionality') can be considered a textbook. It contains the ideas of great mathematicians such as Euclid, Boethius, Sacrobosco, and Fibonacci. A copy of this book came into the hands of Leonardo da Vinci, who invited Pacioli to Milan to collaborate with him on a number of projects, including *De ludo scachorum* ('On the game of chess'), which deals with chess and was only found in 2006 (hidden among 22,000 other volumes in the library of Palazzo Coronini Cronberg in Gorizia, Italy), and *De viribus quantitatis*, whose title could be translated as 'On the power of numbers'. This book can be considered the first work of recreational mathematics in history: it contains puzzles, riddles, and logical challenges. Some of the material is borrowed from other authors, while some is original; it is also the first literary description of a «magic» game played with cards (Bossi, 2008). The manuscript is kept at the University of Bologna and the book was never printed.

In fact, throughout his work, Pacioli proposed problems that could be considered as recreational mathematics. In *Summa*... he presented a version of the «interrupted game problem», which is often cited as the beginning of probability theory given that the Chevalier de Méré posed it to Blaise Pascal, and Pascal discussed the problem in correspondence with Pierre de Fermat. According to the translation in Martín Pliego and Santos del Cerro (2000), the problem can be put as follows: a group of people play a ball game until they reach 60 points. Each game is worth 10 points and they wager a total of 10 ducats. An accident prevents them from finishing the game. One side is left with 50 points and the other with 20. The question is: what proportion of the stakes is due to be paid to each party?

Many more problems, as well as descriptions of mechanical puzzles, can be found in *De viribus quantitatis* (for a detailed analysis of this text, see the master's degree thesis of Tiago Hirth, 2015). Note that this book also mentioned the puzzle we know today as «Chinese rings» or «Cardano's rings» (Figure 1), which can be played online,[1] although we strongly recommend using a real, tangible puzzle. This game is not just for entertainment. Its solution can be formalised using Gray codes (a binary code in which two successive numbers differ in only one of their digits).

## RECREATIONAL MATHEMATICS DURING THE SCIENTIFIC REVOLUTION

The length of this paper forces me to omit references to many authors, but I must mention Gaspard Bachet de Méziriac (1581–1638), who did important work in popularising mathematics. The anecdote that Pierre de Fermat wrote his famous conjecture in the margin of a book, Diophantus' *Arithmetica*, is well known;

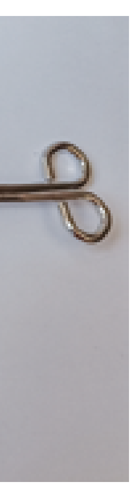
[1] https://blogs.mathworks.com/cleve/2017/02/06/patience-chinese-rings-puzzle

Figure 1. A version of the Chinese rings or Cardano's rings puzzle, consisting of several hooks attached to a base. A steel ring is attached to each hook and a long, movable piece is initially threaded around the nails and through the rings. The goal is to set that piece free.

however, fewer people know that he did so in Bachet's edition! The latter's most important legacy in terms of recreational mathematics was his book *Problèmes plaisants et delectables qui se font par les nombres* ('Pleasant and delectable problems that are done by numbers'; Bachet, 1624). In this book, Bachet gives methods for constructing magic squares (square matrices in which the numbers contained in each row and column add up to a fixed sum) and poses some curious (but not always original) problems, such as the «Josephus problem», previously mentioned by Ambrose of Milan in 370 AD (Petković, 2009). Josephus was a Jewish historian. In his book *The Jewish war*, he recounts how one day he and 40 other Jews found themselves trapped in a cave surrounded by Romans. They preferred to collectively kill themselves rather than be captured and so they decided to line up in a circle and set one of them as the start of a count. They killed the third, then the sixth, and ninth person, and so on, counting in threes, always killing the third. They also decided that the process would continue until there was only one survivor, who would then have to kill themselves. It seems that Josephus quickly worked out where to stand in the circle so that he would be the last to die. Left alone, he decided to join his Roman captors, settled in Rome and became known as Flavius Josephus.

Bachet offered another version of the problem: 15 Christians and 15 Turks are on a ship and a storm breaks out. The ship's load must be lightened, so they decide to form a circle, start counting and when they reach every ninth person, throw them overboard. They continue to count, starting with the one next to the last person drowned, and when they reach the ninth, they throw them overboard too. We must choose the placement so that all 15 in the same group are saved.

Solving one of these problems provides good entertainment and is also a source of examples of various mathematical techniques and concepts including combinatorics, algorithms, and number theory. Indeed, these techniques are repeated from time to time in research articles, with the most recent one (Hua et al., 2019) applying these algorithms to image encryption. Another application of the Josephus problem from the world of card manipulation is known as the «Australian shuffle» (Alegria, 2012).



Fernando Blasco

Many others used Bachet's book to write their own mathematical amusements. Among them was Jacques Ozanam (1640–1717), who published *Récréations mathématiques* ('Mathematical recreations'; Figure 2). The book became a real reference because after Ozanam, Jean-Étienne Montucla (1725–1799) continued to edit and improve it and the English translations were quite intriguing, as we will see a little later.

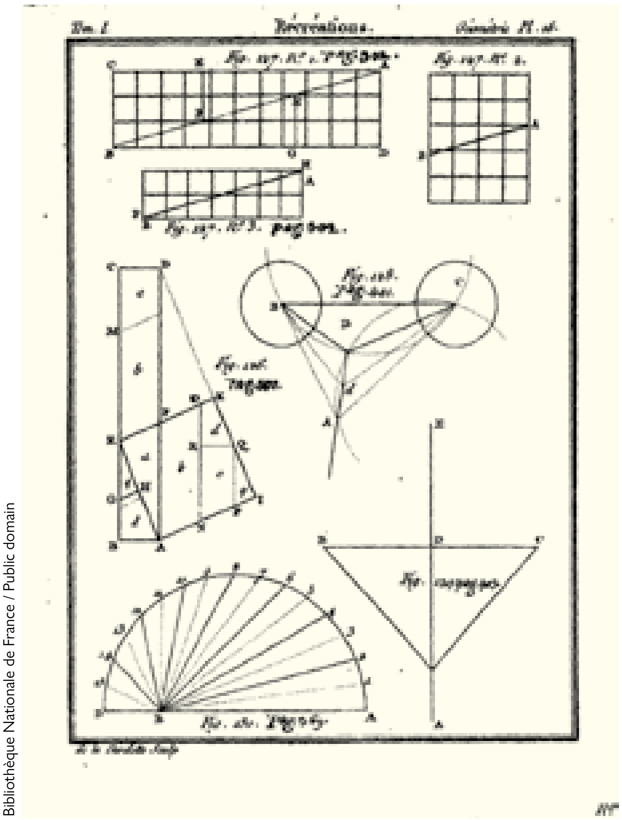


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Figure 2. Image from the book *Récréations mathématiques et physiques* by Ozanam (1778, Figure 15), which presents geometric exercises such as calculating the extension of an area or dissection of a rectangle to a square.

## EULER'S CONTRIBUTIONS TO RECREATIONAL MATHEMATICS

Leonhard Euler (1707–1783) is considered one of the greatest mathematicians in history and is credited with contributing to the development of many mathematical ideas and techniques. He was also very important in the field of recreational mathematics in the 18th century – whether consciously or unconsciously, we do not know. In any case, one of the problems that is still often posed as a mathematical pastime today is «The seven bridges of Königsberg».

The mathematicians Carl Gottlieb Ehler and Heinrich Kuhn wanted to promote mathematical knowledge in the Prussian Empire and the town of Könisberg had a very special geographical feature: the river Pregel had two islands and the town was divided into five parts that could be connected by seven bridges. In one of his letters to Euler, Ehler described the problem of travelling through the five parts of the town by crossing each bridge only once. Euler found that there was no solution to this problem and in doing

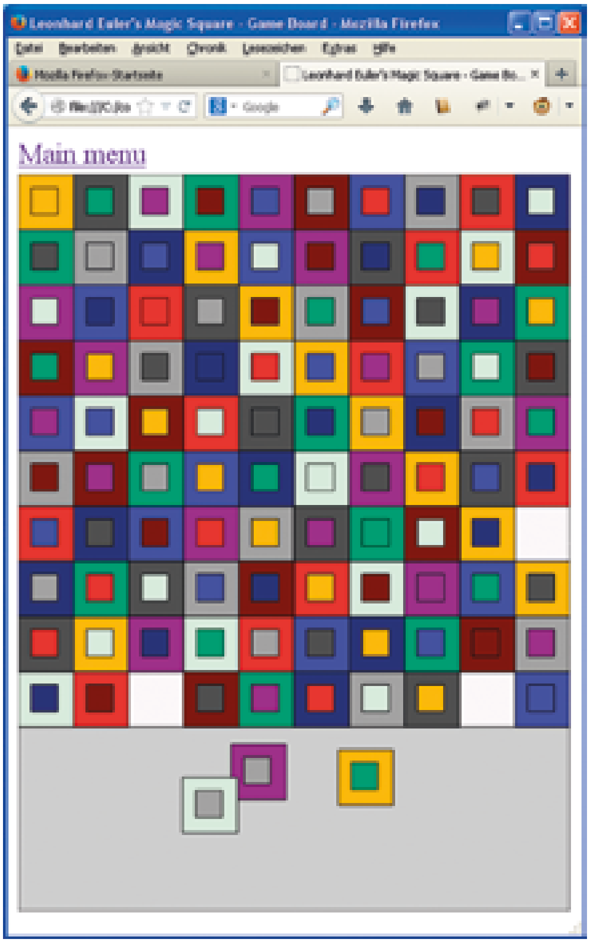





Figure 3. Online simulator for solving Greco-Latin squares, by Oliver Merkel. The picture shows a possible solution for a 10×10 square.

so, launched two mathematical disciplines: topology and graph theory. The Könisberg bridge problem, and in general, all problems involving graphs, are also classics of recreational mathematics, as are maze problems or «The travelling salesperson» problem: in the knowledge of the distances between each pair of cities to be visited, one has to find the shortest route that passes through each city exactly once and returns to the starting point.

Another problem proposed by Ozanam was to draw the ace, jack, queen, and king of each of the four suits from a French deck and arrange them so that each row and column contained one card of each value and suit. In 1782, Euler wrote an article entitled *Recherches sur une nouvelle espèce de quarrés magiques* ('Research on a new species of magic squares') in which he posed a problem similar to Ozanam's, albeit a little more complicated. As with the previous examples, I would like to ask the reader to take some time to consider the solution to the following problem. Six different regiments have six officers, each with a different rank. Can these 36 officers be arranged in a square so that each row and column contains one officer of each rank and one person from each regiment?

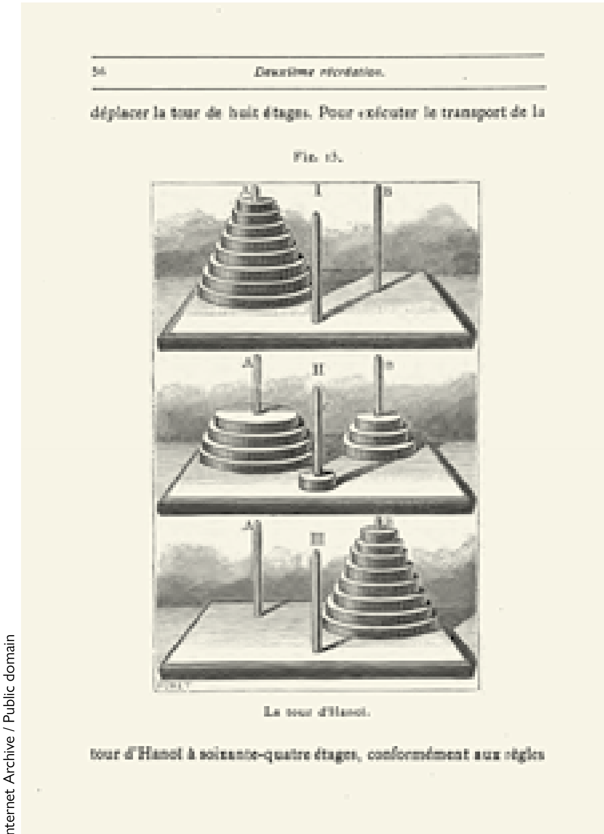
Euler (1782/2007) studied which square sizes of similar puzzles could be solved and concluded that there is no solution if the square size is an even number that is not a multiple of 4. This would imply that neither the 6×6 nor the 10×10 problem could be solved. In 1901, the amateur mathematician Gaston Tarry wrote down all the possible cases and proved that the officers' problem was indeed unsolvable. However, Euler was wrong in his assumption, because in 1959, Ernest Tilden Parker, a professor at the University of Illinois (USA), found a solution for the 10×10 size using a UNIVAC computer (making it one of the first combinatorics problems to be solved on a digital computer). In fact, Parker himself, together with Raj Chandra Bose and Sharadchandra Shankar Shrikhande, also experts in the field, proved that the only sizes for which there are no such Greco-Latin squares, as we now call them (Figure 3), are $n = 2$ and $n = 6$. Of note, today, Greco-Latin squares are used in agricultural and forestry experiments to study the behaviour of different plant species under different phytosanitary treatments.

## POPULARISING RECREATIONAL MATHEMATICS

Sudoku became popular in the press in the year 2000. A Sudoku is a puzzle game played on a square divided into smaller sub-squares. The aim of the game is to fill in all the cells of the square with numbers from 1 to 9 so that each row, column, and 3×3 sub-square contains all the numbers from 1 to 9 without repetition. The appearance and success of these mathematical problems in the press was a surprising phenomenon, although there had been an earlier tradition of presenting mathematical problems in publications aimed at all audiences. An important example was the *Ladies' Diary*, published annually in London between 1704 and 1841 (today it would be considered politically incorrect to introduce it, as it once was, with the phrase «containing new improvements in arts and sciences, and many entertaining particulars: designed for the use and diversion of the fair sex»). The publication contained, among other things, problems in the form of poems, geometrical challenges, and astronomical questions. We should also bear in mind that the population at that time had much more practical astronomical knowledge than we do today given that this knowledge helped them to find their way and tell the time.

The following example was question 220 in issue 38 of the almanac and can be found in Charles Hutton's (1737–1823) compilation of problems and solutions that appeared in the *Ladies' Diary* journal (which he edited between 1774 and 1817): Being at sea on the first of May, on a clear morning, two observations of the sun are made and the difference in elevation is found to be 16° 30', the difference in azimuth to be 34°, and the difference in time to be 2 h 15 m. The problem asks for the latitude and time of day.

Hutton published several books on recreational science; in 1803 he translated and published Montucla's edition of Ozanam's book (Hutton, 1844) which we have already noted was an inspiration to many authors and had many editions. In 1840 Edward Riddle (1788–1854) published a new edition and included a game: trying to guess a number by showing lists full of numbers and asking which of them contained the number the reader had thought of. This game is now a classic of mathematical magic (part of recreational mathematics), and its basis is the binary number system. This source is sometimes cited as the first to contain a description of the game, but it had already been published in Spain a year earlier. The naturalist Juan Mieg (1780–1859) came to Spain with Ferdinand VII to found the School of Physics and Chemistry in the Royal Palace in Madrid. The first chapter of his dissemination book (Mieg, 1839) was devoted to «arithmetical and geometrical chances» and included the game described above. The game was popularised in 1880 – also in Spain – in the form of an advertising brochure used to spread the word about leisure science.




Internet Archive / Public domain

Figure 4. Image from a book by Édouard Lucas (Lucas, 1891, p. 56), which presents the Tower of Hanoi problem. The task is to move a tower of discs of different sizes, stacked from largest to smallest in diameter, from one pole to another using a third pole as an auxiliary space, by following certain rules: only one disc can be moved at a time, only the top disc of a stack can be moved, and a large disc cannot be placed on top of a smaller one.

In France, Édouard Lucas (1842–1891) was simultaneously writing the four volumes of *Récréations mathématiques* (Lucas, 1891), which included original problems such as the well-known «Tower of Hanoi puzzle» (Figure 4; which, again, can be played online,[2] although we recommend playing it with physical pieces); Lucas also mathematically solved the Cardano's rings (or Chinese rings) problem. Today, the Tower of Hanoi puzzle is often used to explain recursion when teaching programming.

**«One of the problems that is still often posed as a mathematical pastime today is "The seven bridges of Königsberg"»**

However, at the beginning of the 20th century, the real architects of the dissemination of mathematics in the press were Henry Ernest Dudeney (1857–1930) in Great Britain and Sam Loyd (1841–1911) in the United States, whose puzzles were almost always accompanied by a story to put the problem in context. Their books are now in the public domain (Dudeney, 1908; Loyd, 1914) and I recommend that the reader try to solve some of the entertaining problems they suggest. Although they were on different sides of the Atlantic, they were aware of each other's work and sometimes even copied each other's ideas for their own problems. Dudeney dissections – geometric dissections of one set of figures into others – are well known; an example is his renowned decomposition of the square (Figure 5). The Wallace–Bolyai–Gerwien theorem states that, given two polygons of the same area, one can be cut into a finite number of pieces so that, when repositioned, they give rise to the other. However, this theorem cannot be extended to three-dimensional space (Klee & Wagon, 1991).

[2] http://towersofhanoi.info/Play.aspx

Phidauex / Wikimedia Commons

Figure 5. Recreation of a Dudeney dissection; in this case, the decomposition of a square results in a triangle.

**«In the second half of the 20th century, and in particular thanks to the work of Martin Gardner, recreational mathematics began to grow»**

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Figure 6. The Gingerbread problem in *Puzzleland*, created by the American recreational mathematician Sam Loyd (1914, p. 285). This puzzle challenges the reader to cut an irregularly shaped cake into square portions.

Loyd also used dissections in his puzzles, although perhaps in a less formal way. For example, in the Gingerbread Puzzle (Loyd, 1914, p. 285) the question is how to cut an irregularly shaped cake into square pieces for sale: the cut must be made along the lines of these portions to divide the cake into two pieces that together form an 8×8 square. The second challenge of the problem is to divide the cake into two pieces that are as large as possible and of the same size and shape (even if one must be turned over and put on top of the other).

Elsewhere in Eastern Europe, the science populariser Yakov Perelman (1882–1942) wrote books on recreational science, selling more than 15 million copies in Russia. His books were translated into local languages throughout Europe and reached Spain through Mir Publishers, which published translations of the Russian originals at very affordable prices. As far as mathematics is concerned, we know of his books *Algebra can be fun*, *Geometry for entertainment*, *Mathematics can be fun,* and *Arithmetic for entertainment*. Like Dudeney and Loyd, Perelman accompanied mathematical problems with stories to put them into context. In many cases the problems were similar to those of the other aforementioned authors, although Perelman's work not only suggests activities, but also included historical facts and the mathematics behind each problem.

In the second half of the 20th century, and in particular thanks to the work of Martin Gardner (1914–2010), recreational mathematics began to grow. He «turned dozens of innocent youngsters into math professors and thousands of math professors into innocent youngsters», in the words of Persi Diaconis (Gardner, 2001), another mathematician who did not neglect recreational mathematics. Gardner published more than 90 books and although all of them are worthwhile, purely for sentimental reasons I would like to highlight *Mathematical carnival* (Gardner, 1980), my first contact with recreational mathematics. The book also approaches a possible definition of recreational mathematics. In his own words:

> The line between entertaining math and serious math is a blurry one. Many professional mathematicians regard their work as a form of play, in the same way professional golfers or basketball stars might. In general, math is considered recreational if it has a playful aspect that can be understood and appreciated by nonmathematicians. Recreational math includes elementary problems with elegant, and at times surprising, solutions. It also encompasses mind-bending paradoxes, ingenious games, bewildering magic tricks and topological curiosities such as Möbius bands and Klein bottles. (Gardner, 1998)

For 25 years he wrote the «Mathematical games» column for *Scientific American*. For me, one of the entries, on John Conway's game «life», stands out in particular. He also wrote about the public-key cryptosystem developed by Ronald Rivest, Adi Shamir, and Leonard Adelman; popularised the work of Escher;


Eoin Gill (Maths Week Ireland, SETU)

Figure 7. David Singmaster solving a ring and chain puzzle in Dublin in 2011.

showed the creations of Piet Hein (such as the popular Soma cube); and wrote about the mathematics in the work of Salvador Dalí. His column also constantly referred to Lewis Carroll, who had been another fan of recreational mathematics. He wrote about Ernö Rubik and his cube, although it was David Singmaster (1939–2023) who succeeded in systematising the solution. In fact, Singmaster (Figure 7) should be recognised as the person who first delved into the history of recreational mathematics. He was a scholar and a collector of old books, games, and mechanical puzzles. Throughout his academic life he compiled a chronology of recreational mathematics that can be found on the internet but has also been published in two volumes (Singmaster, 2021). Despite all the above, many other references and the work of several other authors are missing if we want to address the progress of recreational mathematics in the 20th and 21st centuries, as characterised by the emergence of social networks and new methods of dissemination. However, that is the subject of another monographic text.

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